MATH2101 Complex Analysis (Year 2008/09)
Examination questions and solutions

1. (a) What does it mean for a function f to be holomorphic in the domain $\Omega \subset \mathbb{C}$?

   (b) Describe three types of isolated singularities of a function f by explaining how they are related to the principal part of its Laurent expansion.

   (c) Let f be an entire function, satisfying the inequality $|f(z)| \leq C\sqrt{|z|}$ for all $z \in \mathbb{C}$ with some positive constant C. Prove that $f(z) = 0$ for all $z \in \mathbb{C}$.

Solution.

(a) A function f is said to be holomorphic in a domain Ω if it is differentiable at every point of Ω.

(b) If a function f has an isolated singularity at a point z_0, then it has the Laurent expansion of the form

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n.$$

If $c_k = 0$ for $k < -M, M > 0$ and $c_{-M} \neq 0$, then the function is said to have a pole of order M.

If there is no such number $N \in \mathbb{Z}$ that $c_k = 0$ for all $k < N$, then the singularity is said to be essential.

If $c_k = 0$ for all $k < 0$, then the singularity is said to be removable.

(c) Since f is entire, it can be expanded in the Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with the infinite radius of convergence. Here

$$a_n = \frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)}{z^{n+1}} dz$$

for arbitrary $R > 0$. Estimate the integral using the estimation result, established in the lectures:

$$|a_n| \leq \frac{1}{2\pi} \max_{|z|=R} \left(|f(z)||z|^{-n-1}\right) 2\pi R \leq C R^{\frac{1}{2}-n}.$$

For $n \geq 1$ the right hand side tends to zero as $R \to \infty$, so that $a_n = 0$. If $n = 0$, then the right hand side tends to zero as $R \to 0$, so that $a_0 = 0$ as well. This shows that $f(z) = 0$ for all $z \in \mathbb{C}$.

2. (a) Suppose that a function f is holomorphic on the disk $D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$ with some $r > 0$, and that it has a zero of order m at z_0. Show that the function

$$h(z) = \frac{f'(z)}{f(z)} \tag{1}$$

has a simple pole at z_0 and that $Res(h, z_0) = m$.

(b) Suppose that a function f is holomorphic on the punctured disk $D'(z_0, r) = \{z \in \mathbb{C} : 0 < |z - z_0| < r\}$ with some $r > 0$, and that it has a pole of order l at z_0. Show that the function $h(z)$ defined in (1) has a simple pole at z_0 and that $Res(h, z_0) = -l$.

(c) Let $p(z)$ be a polynomial of degree n, and let $R > 0$ be a number, such that the disk $D(0, R) = \{z \in \mathbb{C} : |z| < R\}$ contains all roots of $p(z)$. Let $\gamma(0, R)$ be the circular contour of radius R, with the counterclockwise orientation. Using Part (a), compute the integral

$$\int_{\gamma(0,R)} \frac{p'(z)}{p(z)} dz.$$

Solution.

(a) By definition of f:

$$f(z) = \sum_{k=m+1}^{\infty} c_k (z - z_0)^k, c_{m+1} \neq 0,$$

and hence,

$$f(z) = (z - z_0)^m g(z),$$

with a function g, holomorphic on the disk, such that $g(z_0) \neq 0$. Therefore $f'(z) = m(z - z_0)^{m-1} g(z) + (z - z_0)^m g'(z)$, and

$$h(z) = \frac{m}{z - z_0} + \frac{g'(z)}{g(z)}.$$

The second summand is analytic in a neighbourhood of z_0, so that indeed, h has at z_0 a simple pole with the residue m.

(b) By definition of f:

$$f(z) = \sum_{k=-l}^{\infty} c_k (z - z_0)^k, c_{-l} \neq 0,$$

and hence,

$$f(z) = (z - z_0)^{-l} g(z),$$

with a function g, holomorphic on the disk, such that $g(z_0) \neq 0$. Therefore $f'(z) = -l(z - z_0)^{-l-1} g(z) + (z - z_0)^{-l} g'(z)$, and

$$h(z) = -\frac{l}{z - z_0} + \frac{g'(z)}{g(z)}.$$

The second summand is analytic in a neighbourhood of z_0, so that indeed, h has at z_0 a simple pole with the residue $-l$.

(c) The polynomial p has exactly n roots (counting multiplicity). Let $z_1, z_2, \ldots, z_s, s \leq n$, be the roots of p with multiplicities $m_1, m_2, \ldots, m_s$, so that $m_1 + m_2 + \cdots + m_s = n$. By Part (a), this means that the function $h = p'p^{-1}$ is analytic on the disk $D(0, R)$ except at the simple poles at the points $z_1, z_2, \ldots, z_s$, and $Res(h, z_j) = m_j, j = 1, 2, \ldots, s$. By Cauchy's Residue Theorem,

$$\int_{\gamma(0,R)} h(z)dz = 2\pi i \sum_{j=1}^{s} Res(h, z_j) = 2\pi i(m_1 + m_2 + \cdots + m_s) = 2\pi i n.$$

3. (a) Find all poles of the function

$$f(z) = \frac{1}{z(e^z - 1)}.$$

Determine the order and calculate the residue at each pole.

(b) Using the Cauchy Residue Theorem evaluate the integral

$$I = \int_\Gamma \frac{1}{(z+1)\sin z} dz$$

along the positively oriented circular contour Γ of radius 2, centered at $z_0 = 0$.

Solution.

(a) The denominator has roots at $z = 0$ and at the roots of $e^z - 1$, i.e. at the points $2\pi i n$, $n \in \mathbb{Z}$.

Let us determine the order of the pole at $z = 0$. Writing Taylor's expansion for e^z, we find that

$$e^z - 1 = \sum_{k=1}^{\infty} \frac{z^k}{k!},$$

so that $z = 0$ is a pole of order 2 of the function f. To find the residue use the formula

$$Res(f, 0) = \lim_{z\to 0} \frac{d}{dz} \frac{z}{e^z - 1} = \lim_{z\to 0} \frac{e^z - 1 - ze^z}{(e^z - 1)^2}$$

$$= \lim_{z\to 0} \frac{-ze^z}{2e^z(e^z - 1)} = -\frac{1}{2} \lim_{z\to 0} \frac{z}{e^z - 1} = -\frac{1}{2}.$$

The limit above was found using l'Hôpital's rule.

Let us find the residues at the remaining poles. Note, first of all, that

$$e^z = e^{z - 2\pi i n} = \sum_{k=0}^{\infty} \frac{(z - 2\pi i n)^k}{k!},$$

so the poles at $2\pi i n, n \neq 0$ are simple. The residues are

$$Res(f, 2\pi i n) = \lim_{z\to 2\pi i n} \frac{(z - 2\pi i n)}{z(e^z - 1)} = \frac{1}{2\pi i n} \lim_{z\to 2\pi i n} \frac{(z - 2\pi i n)}{e^z - 1}$$

$$= \frac{1}{2\pi i n} \lim_{z\to 2\pi i n} \frac{1}{e^z} = \frac{1}{2\pi i n}.$$

Again, we have used l'Hôpital's rule.

(b) It is clear that $z_1 = -1$ is a simple pole of the function

$$f(z) = \frac{1}{(z+1)\sin z}.$$

Furthermore, the function $\sin z$ has one root $z_0 = 0$ inside the contour, and

$$\sin z = z - \frac{z^3}{3!} + \dots,$$

so z_0 is a simple pole of f.
Thus, by the Cauchy residue theorem we conclude that

$$I = 2\pi i Res(f,0) + 2\pi i Res(f,-1).$$

Calculate:

$$Res(f,0) = \lim_{z\to 0} \frac{z}{(z+1)\sin z} = \lim_{z\to 0} \frac{z}{\sin z} = 1,$$

$$Res(f,-1) = \lim_{z\to -1} \frac{1}{\sin z} = -\frac{1}{\sin 1}.$$

Consequently,

$$I = 2\pi i \left(1 - \frac{1}{\sin 1}\right).$$

4. (a) Let $f(z) = u(x, y) + iv(x, y)$ be an entire function, and let u be a function of x alone. Show that $f(z) = az + b$ with some constants $a \in \mathbb{R}$ and $b \in \mathbb{C}$.

   (b) Evaluate the following integral by integrating around a suitable closed contour:

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + x + 1} dx.$$

Solution.

(a) By Cauchy-Riemann Equations, $v_x = -u_y = 0$, so that v depends only on y, i.e. $u(x, y) = g(x)$ and $v(x, y) = h(y)$. Using Cauchy-Riemann Equations again, we get $u_x = v_y$, i.e. $g_x(x) = h_y(y)$. This implies that both g_x and h_y are constant functions, and hence

$$g(x) = ax + c_1, h(y) = ay + c_2$$

with some real constants a, c_1, c_2. Now,

$$f(z) = g(x) + ih(y) = az + b, b = c_1 + ic_2,$$

as claimed.

(b) Let

$$g(z) = \frac{e^{iz}}{z^2 + z + 1}.$$

This function has two poles: at $-1/2 \pm i\sqrt{3}/2$.
Define the contour

$$\Gamma^{(R)} = \Gamma_1^{(R)} \cup \Gamma_2^{(R)},$$
$$\Gamma_1^{(R)} = \{\operatorname{Im} z = 0,\ |\operatorname{Re} z| \le R\},\ \ \Gamma_2^{(R)} = \{z = Re^{i\theta},\ \theta \in [0, \pi]\}.$$

Only the pole $z_0 = -1/2 + i\sqrt{3}/2$ is inside $\Gamma^{(R)}$, so, by Cauchy's Residue Theorem,

$$\int_{\Gamma^{(R)}} g(z)dz = 2\pi i Res(g, z_0).$$

Find the residue:

$$Res(g, z_0) = \lim_{z\to z_0}(z - z_0)g(z) = \lim_{z\to z_0}\frac{e^{iz}}{z + 1/2 + i\sqrt{3}/2}$$
$$= \frac{1}{i\sqrt{3}}e^{-\sqrt{3}/2-i/2}.$$

Consequently,

$$\int_{\Gamma^{(R)}} f(z)dz = \frac{2\pi i}{i\sqrt{3}}e^{-\sqrt{3}/2-i/2} = \frac{2\pi}{\sqrt{3}}e^{-\sqrt{3}/2}(\cos 1/2 - i\sin 1/2).$$

Estimate the integral along $\Gamma_2^{(R)}$. By Jordan's Lemma

$$\left|\int_{\Gamma_2^{(R)}} g(z)dz\right| \to 0, R\to\infty.$$

Thus

$$\int_{\mathbf{R}}\frac{\cos x}{x^2+x+1}dx = \mathrm{Re}\int_{\mathbf{R}} g(x)dx = \mathrm{Re}\left(\lim_{R\to\infty}\int_{\Gamma^{(R)}} g(z)dz\right) = \frac{2\pi}{\sqrt{3}}e^{-\sqrt{3}/2}\cos\frac{1}{2}.$$

5. (a) Prove that the function $g(z) = \dfrac{z}{\sin z}$ has a removable singularity at $z = 0$, and $g(z)$ is analytic in a neighbourhood of $z = 0$ if one assumes $g(0) = 1$.

(b) Evaluate the following integral by integrating around a suitable closed contour:

$$\int_{-\infty}^{\infty} \frac{1}{x^4+4} dx.$$

Solution.

(a) Expanding $\sin z$ in the series, we see that

$$g(z) = \frac{z}{\sin z} = \frac{z}{z - \frac{z^3}{6} + \dots} = \frac{1}{1 - \frac{z^2}{6} + \dots} = 1 + \frac{z^2}{6} + \dots.$$

Hence, g has a removable singularity at $z = 0$, and g is analytic if one defines $g(0) = 1$.

(b) Let

$$f(z) = \frac{1}{z^4+4}.$$

This function has four simple poles: at $z = \pm\sqrt{2}e^{i\pi/4}$ and $\pm\sqrt{2}ie^{i\pi/4}$.
Define the contour

$$\Gamma^{(R)} = \Gamma_1^{(R)} \cup \Gamma_2^{(R)},$$
$$\Gamma_1^{(R)} = \{\operatorname{Im} z = 0,\ |\operatorname{Re} z| \le R\}, \quad \Gamma_2^{(R)} = \{z = Re^{i\theta},\ \theta \in [0,\pi]\}.$$

Only the poles $z_1 = \sqrt{2}e^{i\pi/4}$ and $z_2 = \sqrt{2}ie^{i\pi/4}$ are inside $\Gamma^{(R)}$, so, by Cauchy's Residue Theorem,

$$\int_{\Gamma^{(R)}} f(z)dz = 2\pi i\big(Res(f,z_1) + Res(f,z_2)\big).$$

Find the residues:

$$Res(f,z_1) = \lim_{z\to z_1}(z-z_1)f(z) = \lim_{z\to z_1} \frac{1}{(z+\sqrt{2}e^{i\pi/4})(z-\sqrt{2}ie^{i\pi/4})(z+\sqrt{2}ie^{i\pi/4})}$$

$$= \frac{1}{2\sqrt{2}e^{i\pi/4}\cdot\sqrt{2}e^{i\pi/4}(1-i)\cdot\sqrt{2}e^{i\pi/4}(1+i)} = -\frac{\sqrt{2}}{16}e^{i\pi/4},$$

$$Res(f, z_2) = \lim_{z \to z_2} (z - z_2) f(z) = \lim_{z \to z_2} \frac{1}{(z + \sqrt{2}e^{i\pi/4})(z - \sqrt{2}e^{i\pi/4})(z + \sqrt{2}ie^{i\pi/4})}$$

$$= \frac{1}{\sqrt{2}e^{i\pi/4}(1+i) \cdot \sqrt{2}e^{i\pi/4}(i-1) \cdot 2i\sqrt{2}e^{i\pi/4}} = \frac{i\sqrt{2}}{16}e^{-3i\pi/4} = \frac{\sqrt{2}}{16}e^{-i\pi/4}.$$

Consequently,

$$\int_{\Gamma(R)} f(z)dz = -\frac{2\pi i\sqrt{2}}{16}(e^{i\pi/4} - e^{-i\pi/4}) = \frac{\pi\sqrt{2}}{4}\sin\frac{\pi}{4} = \frac{\pi}{4}.$$

Estimate the integral along $\Gamma_2^{(R)}$:

$$\left|\int_{\Gamma_2^{(R)}} f(z)dz\right| \leq \max_{z \in \Gamma_2^{(R)}} |f(z)| \cdot 2\pi R \leq \frac{1}{R^4 - 1} \cdot 2\pi R \to 0,$$

as $R \to \infty$. Thus

$$\int_{\mathbf{R}} f(x)dx = \lim_{R \to \infty} \int_{\Gamma(R)} f(z)dz = \frac{\pi}{4}.$$

6. (a) Find the Laurent expansions of the function

$$g(z) = \frac{1}{(z+1)(z-3)},$$

valid for

i. $1 < |z| < 3$;

ii. $1 < |z-2| < 3$.

(b) Prove that

$$\int_0^{2\pi} \frac{\cos 2\theta}{2+\cos\theta} d\theta = 2\pi\left(-4 + \frac{7}{\sqrt{3}}\right),$$

using the Cauchy Residue Theorem.

Solution.

(a) Expand, using partial fractions:

$$g(z) = -\frac{1}{4(z+1)} + \frac{1}{4(z-3)}.$$

i. Re-write the above expansion:

$$\begin{aligned} g(z) = & -\frac{1}{4z(1+\frac{1}{z})} - \frac{1}{12(1-\frac{z}{3})} = -\frac{1}{4z}\sum_{k=0}^{\infty}\frac{(-1)^k}{z^k} - \frac{1}{12}\sum_{k=0}^{\infty}\frac{z^k}{3^k} \\ = & -\frac{1}{4}\sum_{k=0}^{\infty}\frac{(-1)^k}{z^{k+1}} - \frac{1}{12}\sum_{k=0}^{\infty}\frac{z^k}{3^k}. \end{aligned}$$

ii. Denote $w = z - 2$ and re-write:

$$\begin{aligned} g(z) = & -\frac{1}{4(w+3)} + \frac{1}{4(w-1)} = -\frac{1}{12(1+\frac{w}{3})} + \frac{1}{4w(1-\frac{1}{w})} \\ = & -\frac{1}{12}\sum_{k=0}^{\infty}(-1)^k\frac{w^k}{3^k} + \frac{1}{4w}\sum_{k=0}^{\infty}\frac{1}{w^k} \\ = & -\frac{1}{12}\sum_{k=0}^{\infty}(-1)^k\frac{(z-2)^k}{3^k} + \frac{1}{4}\sum_{k=0}^{\infty}\frac{1}{(z-2)^{k+1}}. \end{aligned}$$

(b) Make the substitution: $z = e^{i\theta}$, so that

$$\cos\theta = \frac{z + z^{-1}}{2}, \ \cos 2\theta = \frac{z^2 + z^{-2}}{2}, d\theta = -i\frac{dz}{z},$$

whence

$$I = -i\int_\Gamma \frac{\frac{1}{2}(z^2 + z^{-2})}{2 + \frac{1}{2}(z + z^{-1})}\frac{dz}{z} = -i\int_\Gamma \frac{z^4 + 1}{z^2(z^2 + 4z + 1)}dz,$$

where Γ is the positively oriented circular countour of radius 1 centered at $z_0 = 0$. The denominator has three distinct roots:

$$z_0 = 0, z_1 = -2 + \sqrt{3}, z_2 = -2 - \sqrt{3}.$$

Only z_0, z_1 are inside the disk. Thus, by the Cauchy Residue Theorem

$$I = 2\pi Res(f, z_0) + 2\pi Res(f, z_1), \ f(z) = \frac{z^4 + 1}{z^2(z^2 + 4z + 1)}.$$

The pole at $z = 0$ is of order 2, so

$$Res(f, 0) = \lim_{z\to 0}\frac{d}{dz}\frac{z^4 + 1}{z^2 + 4z + 1} = \lim_{z\to 0}\frac{3z^3(z^2 + 4z + 1) - (z^4 + 1)(2z + 4)}{(z^2 + 4z + 1)^2} = -4.$$

The pole at z_1 is simple, and hence

$$\begin{aligned} Res(f, z_1) = \lim_{z\to z_1}(z - z_1)\frac{z^4 + 1}{z^2(z - z_1)(z - z_2)} &= \frac{z_1^4 + 1}{z_1^2(z_1 - z_2)} \\ = \frac{(-2 + \sqrt{3})^4 + 1}{(-2 + \sqrt{3})^2 \cdot 2\sqrt{3}} = \frac{2(49 - 28\sqrt{3})}{(7 - 4\sqrt{3}) \cdot 2\sqrt{3}} &= \frac{7}{\sqrt{3}}. \end{aligned}$$

Therefore

$$I = 2\pi\left(-4 + \frac{7}{\sqrt{3}}\right),$$

as required.